



TITLE:

On circular operators (Prospects of non-commutative analysis in operator theory)

AUTHOR(S):

Ota, Schoichi

CITATION:

Ota, Schoichi. On circular operators (Prospects of non-commutative analysis in operator theory). 数理解析研究所講究録 2010, 1678: 128-133

ISSUE DATE:

2010-04

URL:

<http://hdl.handle.net/2433/141295>

RIGHT:

Circular 作用素について (On circular operators)

九州大学 大学院芸術工学研究院 太田 昇一 (Schôichi Ôta)
Faculty of Design, Kyushu University

稠密な定義域を持つ Hilbert 空間上の作用素に対して、‘circular’なる概念を導入する。この概念は、有界作用素に対して W. Mlak 等によって、作用素論および量子力学の観点から研究されている。本講演では circular 作用素に付随して現れる強連続 1 径数ユニタリー群について考察し、 q -変形作用素族の q -正規作用素がもつ‘性質 Q’との関わりについて述べる。

1 Circularity

A densely defined operator T in a Hilbert space \mathcal{H} is said to be **circular** if T is unitarily equivalent to $e^{it}T$ for all $t \in \mathbb{R}$. Clearly the spectrum of a circular operator is circularly symmetric at origin.

Example 1 Let S be a closed densely defined operator in a separable Hilbert space \mathcal{H} . If there are an orthonormal basis $\{e_n\}$ ($n \in \mathbb{Z}$) and a sequence $\{w_n\}$ ($w_n \neq 0, n \in \mathbb{Z}$) of complex numbers such that

$$\mathcal{D}(S) = \left\{ \sum_{n=-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_n|^2 |w_n|^2 < \infty \right\}$$

and

$$S e_n = w_n e_{n+1}$$

for all $n \in \mathbb{Z}$, then S is called a bilateral (injective) weighted shift with weights $\{w_n\}$ (with respect to $\{e_n\}$). A unilateral weighted shift is defined by the replacement \mathbb{Z} with \mathbb{N} analogously.

Every bilateral or unilateral, weighted shift is circular.

Let us recall irreducibility for a possibly unbounded operator in \mathcal{H} . Let T be a closed densely defined operator in \mathcal{H} . A closed subspace \mathcal{M} of \mathcal{H} is said to reduce T if the following two conditions are satisfied:

1. $P_{\mathcal{M}}\mathcal{D}(T) \subseteq \mathcal{D}(T)$.
2. $T(\mathcal{M} \cap \mathcal{D}(T) \subseteq \mathcal{M}$ and $T(\mathcal{M}^\perp \cap \mathcal{D}(T) \subseteq \mathcal{M}^\perp$.

Here $P_{\mathcal{M}}$ denotes the orthogonal projection onto \mathcal{M} . If there is no non-trivial reducing subspace of T , then T is said to be *irreducible*.

Lemma 2 *Let T be an irreducible, closed densely defined operator in a separable Hilbert space \mathcal{H} . If T is circular, then there are a family $\{U_t\}_{t \in \mathbb{R}}$ of unitary operators on \mathcal{H} and a mapping $m(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{T}$ such that*

1. $U_t T = e^{it} T U_t$ for all $t \in \mathbb{R}$.
2. $U_s U_t = m(s, t) U_{s+t}$, $U_0 = I$ (identity operator) for all $s, t \in \mathbb{R}$.
3. the map $m(\cdot, \cdot)$ satisfies

$$m(s, 0) = m(0, s) = 1 \quad \text{and} \quad m(s + t, u) m(s, t) = m(s, t + u) m(t, u)$$

for $s, t \in \mathbb{R}$.

Here, \mathbb{T} is the multiplicative group of complex numbers with modulus 1.

Moreover, if the above $\{U_t\}$ is so chosen that $t \mapsto U_t$ is measurable, there exists a strongly continuous one-parameter unitary group $\{V_t\}$ satisfying the above condition 1, that is, $V_t T = e^{it} T V_t$ for all $t \in \mathbb{R}$.

2 Strong circularity

Let T be a closed densely defined operator in a Hilbert space \mathcal{H} . If there is a strongly continuous one-parameter unitary group $\{U_t\}_{t \in \mathbb{R}}$ such that

$$U_t T = e^{it} T U_t \quad (t \in \mathbb{R}),$$

then T is said to be **strongly circular** and $\{U_t\}_{t \in \mathbb{R}}$ is called a unitary group associated with T .

Example 3 (Mlak) *If S is the creation operator in a separable Hilbert space, that is, the unilateral weighted shift with weights $\{w_n\}$ given by $w_n = \sqrt{n+1}$ ($n \in \mathbb{N}$), then S is strongly circular.*

Let S be a unilateral or bilateral weighted shift in a separable Hilbert space \mathcal{H} . Then S is strongly circular.

In fact, let S be a bilateral weighted shift in \mathcal{H} with weights $\{w_n\}$ with respect to $\{e_n\}$. Define a closed densely defined operator by

$$\mathcal{D}(N) = \left\{ \sum_{n=-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_n|^2 |n|^2 < \infty \right\}$$

and

$$Ne_n = ne_n \quad (n \in \mathbb{Z}).$$

Then N is self-adjoint, and

$$e^{itN} S e_n = e^{it} S e^{itN} e_n$$

for all $n \in \mathbb{Z}$. It follows that S is a strongly circular operator with the associated unitary group $\{e^{itN}\}$.

For a bounded operator B and a densely defined operator T , $BT \subseteq TB$ means that

$$B\mathcal{D}(T) \subseteq \mathcal{D}(T) \text{ and } BT\eta = TB\eta \quad (\eta \in \mathcal{D}(T)).$$

Lemma 4 *Let S be a densely defined operator in a Hilbert space \mathcal{H} and T be a closed densely defined operator in \mathcal{H} . Let $\{U_t\}_{t \in \mathbb{R}}$ and $\{V_t\}_{t \in \mathbb{R}}$ be strongly continuous one-parameter unitary groups on \mathcal{H} with infinitesimal generators A and B respectively, that is, $U_t = e^{itA}$, $V_t = e^{itB}$. Then the following conditions are equivalent:*

1. For all $t \in \mathbb{R}$,

$$U_t S \subseteq T V_t.$$

2. For all $\lambda \in \mathbb{C}$ with $\Im \lambda \neq 0$,

$$(\lambda - A)^{-1} S \subseteq T (\lambda - B)^{-1}.$$

Theorem 5 *Let T be a closed densely defined operator in a Hilbert space \mathcal{H} . Then T is strongly circular if and only if there is a self-adjoint operator A in \mathcal{H} such that*

$$(\lambda - A)^{-1} T \subseteq T (\lambda - I - A)^{-1} \quad (1)$$

for all $\lambda \in \mathbb{C}$ with $\Im \lambda \neq 0$.

Proof. Suppose T is strongly circular. Then there is a strongly continuous one-parameter unitary group $\{U_t\}_{t \in \mathbb{R}}$ such that $U_t T = e^{it} T U_t$ for all $t \in \mathbb{R}$. Set

$$V_t = e^{it} U_t \quad \text{for each } t \in \mathbb{R}.$$

Then, $\{V_t\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group on \mathcal{H} . Let A be the infinitesimal generator of $\{U_t\}_{t \in \mathbb{R}}$. Then it follows from the semigroup theory that the infinitesimal generator of $\{V_t\}_{t \in \mathbb{R}}$ is $I + A$. Putting $T = S$ in the above, A satisfies relation (1).

Conversely, suppose A is a self-adjoint operator satisfying relation (1). Put

$$U_t = e^{itA} \quad \text{and} \quad V_t = e^{it} e^{itA}$$

for $t \in \mathbb{R}$. Then we obtain $U_t T \subseteq e^{it} T U_t$ for all $t \in \mathbb{R}$. Since each U_t is unitary, $U_t \mathcal{D}(T) = \mathcal{D}(T)$. Hence, $U_t T = e^{it} T U_t$ for all $t \in \mathbb{R}$.

3 q -deformed circularity

Let T be a densely defined operator in a Hilbert space \mathcal{H} . If there is a positive real number q with $q \neq 1$ such that T is unitarily equivalent to qT , then we say that T has property Q .

Proposition 6 . Suppose that a nontrivial closed densely defined operator T has property Q . Then,

1. T is unbounded.
2. The spectrum contains zero.
3. The absolute value $|T|$ has also property Q .

Example 7 Let T be a closed densely defined operator in \mathcal{H} . If T satisfies

$$TT^* = qT^*T \quad (q > 0, q \neq 1),$$

then T is called a q -normal operator. It should be noticed that elements satisfying this relation in a formal algebraic sense appear at various circumstances in the theory of quantum group theory. A non-trivial q -normal operator T is always unbounded and has sufficient large spectrum in the sense of the planar Lebesgue measure. Especially, every q -normal operator T is unitarily equivalent to qT . Thus the class of operators possessing property Q contains all q -normal operators.

Definition 8 Let T be a densely defined operator in \mathcal{H} . If there is a positive real number q with $q \neq 1$ such that T is unitarily equivalent to $qe^{it}T$ for all $t \in \mathbb{R}$, then T is called a q -deformed circular (simply, q -circular) operator.

Circularity may be considered as q tends to 1 in the above. Clearly, a q -circular operator has property Q .

Example 9 (q -circular weighted shifts) If a bilateral weighted shift has property Q , then it is q -circular. Hence, a q -normal bilateral weighted shift is q -circular. Moreover the spectrum of a q -circular weighted shift is equal to the whole complex plain.

Theorem 10 Let T be a closed densely defined operator in a Hilbert space \mathcal{H} . Then T is q -circular if and only if T is circular and has property Q .

Proof. Suppose T is q -circular. Then there is a family $\{U_t\}_{t \in \mathbb{R}}$ of unitary operators on \mathcal{H} such that

$$U_t T = qe^{it} T U_t . \quad (2)$$

for all $t \in \mathbb{R}$. It is clear that T has property Q . Put

$$V_t = U_t U_0^{-1} . \quad (3)$$

for all $t \in \mathbb{R}$. We have by above relation (2)

$$\begin{aligned} V_t T &= q U_t U_0^{-1} T = U_t T U_0 \\ &= q e^{it} T U_t U_0^{-1} = e^{it} T V_t . \end{aligned}$$

Thus T is circular. The converse is easily proved by a simple calculation.

Theorem 11 Let T be a closed densely defined operator in \mathcal{H} with the polar decomposition $T = U |T|$. Then T is q -circular if and only if the following statements hold:

1. There is a unitary operator U_0 on \mathcal{H} that commutes with U and satisfies

$$U_0 |T| = q |T| U_0 . \quad (4)$$

2. There is a family $\{V_t\}_{t \in \mathbb{R}}$ of unitary operators on \mathcal{H} such that

$$V_t U = e^{it} U V_t \quad \text{and} \quad V_t |T| = |T| V_t \quad (5)$$

for all $t \in \mathbb{R}$.

Epecially, if this is the case, U is circular.

参考文献

- [1] W. Arveson, D.W. Hadwin, T.W. Hoover and E. Eugene Kymala, *Circular operators*, Indiana Univ. Math. J., **33**(1984), 583–595.
- [2] W. Mlak, *Notes on circular operators* I, II, III, IV, Univ. Jagell. Acta Math., **29**(1992).
- [3] S. Ôta, *q -deformed circular operators*, Integral Equations and Operator Theory, **54**(2006), 555–569.
- [4] A.L. Shields, *Weighted shift operators and analytic function theory*, Math. Surveys, American Mathematical Society, **13**(1974), Providence Rhode Island.